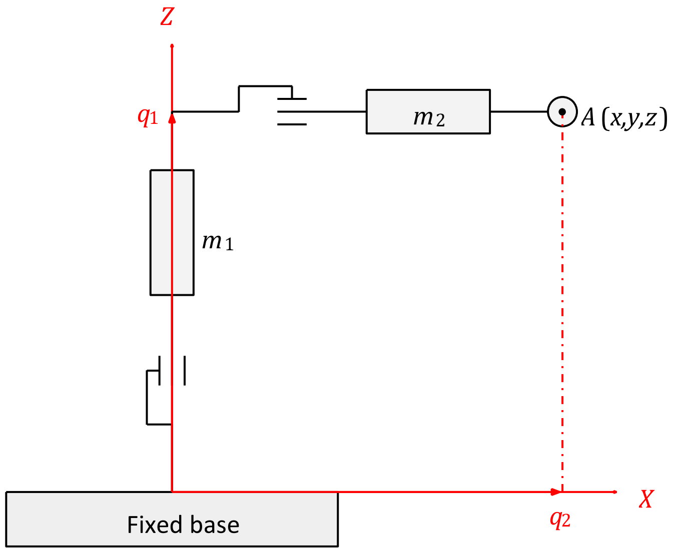
# Exercise set 7 – Dynamics - Solutions

## Exercise 1

*The goal of this exercise is to determine the dynamic model of the Cartesian robot with two axes through the Lagrange approach.*

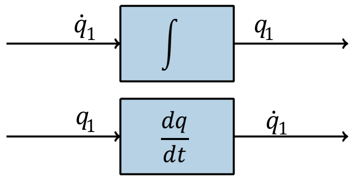



Operational coordinates (tool): $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Joint coordinates: $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

1. Determine the IDM of this Cartesian robot using the Lagrange approach.
2. From the developed model, make conclusions about the inertia (whether the inertia is constant in space), about the coupling of the axes (whether the axes are coupled or decoupled) and about the positional dependence of the controllers (whether the controllers will depend on the configuration/position of the robot).
3. Give the expression of the a priori IDM (function of required torque/forces for a desired trajectory).
4. Determine the DDM.
5. Draw the block representation of the DDM.

   Hint: Example of blocks would be like this:




## Exercise 1 – Solution

*The goal of this exercise is to determine the dynamic model of the two-axis Cartesian robot through the Lagrange approach.*

1. For the IDM, we seek to find the following:

$$\mathbf{\Gamma} = \mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q})$$

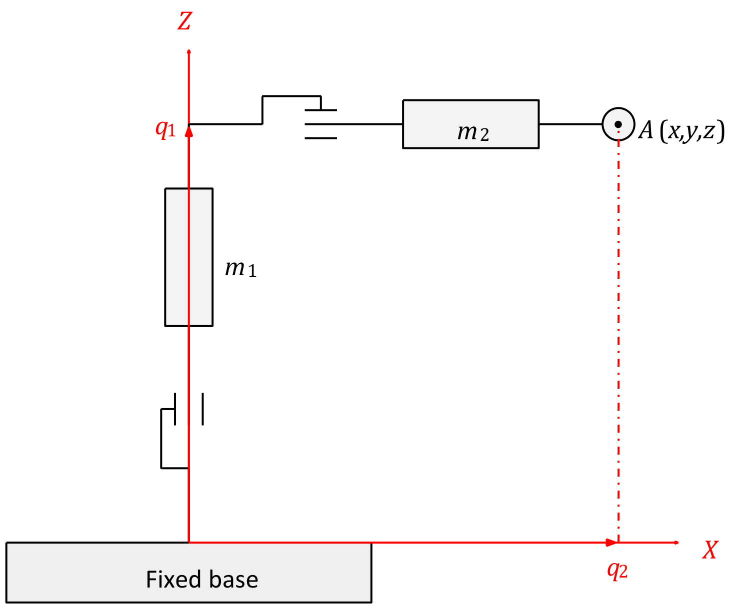
with:

- $\mathbf{\Gamma}$ the generalized torques/joint torques
- $\mathbf{q}$ the generalized coordinates/joint variables
- $\mathbf{B}$ the inertia matrix
- $\mathbf{G}$ the gravity vector
- $\mathbf{C}$ the vector of the centrifugal and Coriolis terms
- $\mathbf{F}$ the friction vector
- $\mathbf{K}$ the rigidity vector

We proceed step by step:

a) Description of the robot:

i. Identify the tool, the joints and the segments of the robot:




ii. Operational coordinates: $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

iii. Joint coordinates: $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

b) Local Jacobians reported to each segment $\mathbf{J}_p^{(i)}$ and $\mathbf{J}_o^{(i)}$

- Segment 1 in translation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_1 \end{bmatrix}$ therefore $\mathbf{J}_p^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

- Segment 2 in translation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} q_2 \\ 0 \\ q_1 \end{bmatrix}$ therefore $\mathbf{J}_p^{(2)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

- Segments 1 and 2 in rotation: $\boldsymbol{J}_o^{(1)} = \boldsymbol{J}_o^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

c) Matrix of inertia $\mathbf{B}(\mathbf{q})$ :

$$\begin{aligned}
\mathbf{B}(\mathbf{q}) &= \sum_{i=1}^{n} \left( m_i \mathbf{J}_p^{(i)T} \mathbf{J}_p^{(i)} + \mathbf{J}_o^{(i)T} \mathbf{R}_i \mathbf{I}_i \mathbf{R}_i^T \mathbf{J}_o^{(i)} \right) \\
&= m_1 \mathbf{J}_p^{(1)T} \mathbf{J}_p^{(1)} + m_2 \mathbf{J}_p^{(2)T} \mathbf{J}_p^{(2)} \\
&= m_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + m_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} m_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_2 & 0 \\ 0 & m_2 \end{bmatrix} \\
&= \begin{bmatrix} (m_1 + m_2) & 0 \\ 0 & m_2 \end{bmatrix}
\end{aligned}$$

d) Vector of gravity $\mathbf{G}(\mathbf{q})$

i. Local gravity terms reported to each segment $g_i(\mathbf{q})$ :

$$g_i(\mathbf{q}) = \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{p,i}^{(j)}$$
$$\mathbf{g}^T = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}$$

- Segment 1:

$$\begin{aligned}
g_1(\mathbf{q}) &= \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{p,1}^{(j)} \\
&= m_1 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + m_2 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= -g\,(m_1 + m_2)
\end{aligned}$$

- Segment 2:

$$\begin{aligned}
g_2(\mathbf{q}) &= \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{p,2}^{(j)} \\
&= m_1 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= 0
\end{aligned}$$

ii. Vector of gravity $\mathbf{G}(\mathbf{q})$ :

$$\begin{aligned}\mathbf{G}(\mathbf{q}) &= \begin{bmatrix} -g_1 \\ -g_2 \\ \vdots \\ -g_n \end{bmatrix} \\ &= \begin{bmatrix} -g_1 \\ -g_2 \end{bmatrix} \\ &= \begin{bmatrix} g(m_1+m_2) \\ 0 \end{bmatrix}\end{aligned}$$

e) Vector of the centrifugal and Coriolis terms: $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathbf{0}$ because there are no centrifugal and Coriolis effects (**B(q)** is constant).

f) Friction vector: $\boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathbf{0}$ because friction is ignored.

g) Stiffness vector: $\mathbf{K}(\mathbf{q}) = \mathbf{0}$ because we consider the robot's segments as infinitely rigid and there is no elasticity in the joints.

h) Finally, the generalized torque $\mathbf{\Gamma}$:

$$\begin{aligned}\mathbf{\Gamma} &= \mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q}) \\ &= \begin{bmatrix} (m_1+m_2) & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} g(m_1+m_2) \\ 0 \end{bmatrix} + \mathbf{0} + \mathbf{0} + \mathbf{0} \\ &= \begin{bmatrix} (m_1+m_2)(\ddot{q}_1+g) \\ m_2\ddot{q}_2 \end{bmatrix}\end{aligned}$$

We therefore have:

$$\mathbf{\Gamma} = \begin{bmatrix} (m_1+m_2)(\ddot{q}_1+g) \\ m_2\ddot{q}_2 \end{bmatrix}$$

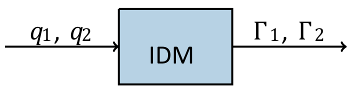
$$\Leftrightarrow \begin{cases} \Gamma_1 = (m_1+m_2)(\ddot{q}_1+g) \\ \Gamma_2 = m_2\ddot{q}_2 \end{cases}$$

2. From the IDM, we deduce that:

- the terms of the inertia matrix are constant (independent of the position).
- the dynamics of axis 1 and axis 2 are completely decoupled.
- the controllers of each axis are independent of the working position $\mathbf{q}$.

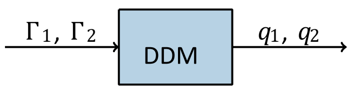
3. A priori IDM:

$$\begin{cases} \Gamma_{1\_\mathrm{ap}} = (m_1+m_2)(\ddot{q}_{1,d}+g) \\ \Gamma_{2\_\mathrm{ap}} = m_2\ddot{q}_{2,d} \end{cases}$$

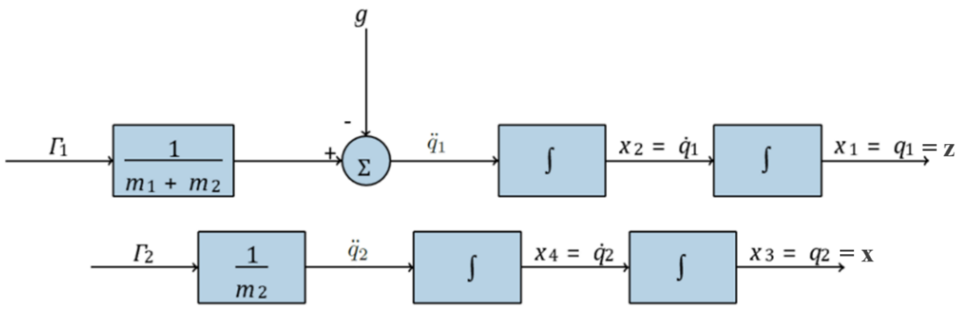



4. DDM:

$$\begin{cases} \ddot{q}_1 & = \dfrac{\Gamma_1}{m_1+m_2} - g \\ \ddot{q}_2 & = \dfrac{\Gamma_2}{m_2} \end{cases}$$



5. Block representation of the DDM:




## Exercise 2

Determine the inverse dynamic model of the two-axis Cartesian robot using the Newton-Euler approach.

## Exercise 2 – Solution

*The goal of this exercise is to determine the IDM of the Cartesian robot with two axes through the Newton-Euler approach.*

We therefore seek:

$$\boldsymbol{\Gamma} = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$$

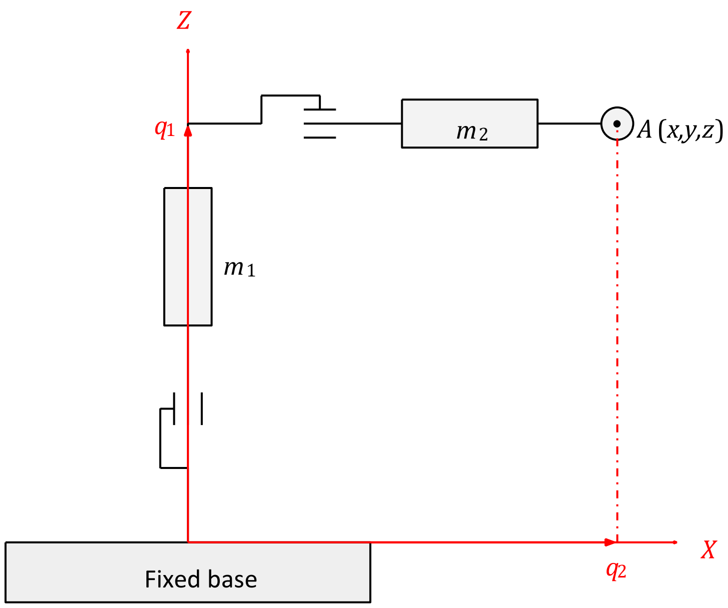
with:

- $\boldsymbol{\Gamma}$ generalized torques/joint torques
- $\boldsymbol{q}$ generalized coordinates/joint variables

We proceed step by step:

1. Description of the robot:

   (a) Identify the tool, the joints and the segments of the robot:



b) Operational coordinates: $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

c) Joint coordinates: $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

2. Newton-Euler equations for each segment:

   (a) Body 0 (fixed base):

$$\mathbf{v_{c0}} = \dot{\mathbf{v}}_{\mathbf{c0}} = \boldsymbol{\omega}_{\mathbf{c0}} = \dot{\boldsymbol{\omega}}_{\mathbf{c0}} = \mathbf{0}$$

Indeed, the base does not move.

   (b) Body 1 (segment 1):

$$\begin{aligned}
\mathbf{v_{c1}} &= \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \\
\dot{\mathbf{v}}_{\mathbf{c1}} &= \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 \end{bmatrix} \\
\boldsymbol{\omega}_{\mathbf{c1}} &= \dot{\boldsymbol{\omega}}_{\mathbf{c1}} = \mathbf{0} \\
\mathbf{f_{01}} &= \begin{bmatrix} f_{x,01} \\ f_{y,01} \\ \Gamma_1 \end{bmatrix} \\
\mathbf{f_{12}} &= \begin{bmatrix} \Gamma_2 \\ f_{y,12} \\ f_{z,12} \end{bmatrix}
\end{aligned}$$

The Newton-Euler equation for segment 1 is:

$$\begin{aligned}
&\mathbf{f_{01}} - \mathbf{f_{12}} + m_1 \left(\mathbf{g} - \dot{\mathbf{v}}_{\mathbf{c1}}\right) = \mathbf{0} \\
\Rightarrow &\begin{bmatrix} f_{x,01} \\ f_{y,01} \\ \Gamma_1 \end{bmatrix} - \begin{bmatrix} \Gamma_2 \\ f_{y,12} \\ f_{z,12} \end{bmatrix} + m_1 \left( \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 \end{bmatrix} \right) = \mathbf{0} \\
\Rightarrow &\begin{cases} \Gamma_2 & = f_{x,01} \\ f_{y,01} & = f_{y,12} \\ \Gamma_1 & = f_{z,12} + m_1(g + \ddot{q}_1) \end{cases}
\end{aligned}$$

c) Body 2 (segment 2):

$$\mathbf{v_{c2}} = \begin{bmatrix} \dot{q}_2 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$
$$\dot{\mathbf{v}}_{\mathbf{c2}} = \begin{bmatrix} \ddot{q}_2 \\ 0 \\ \ddot{q}_1 \end{bmatrix}$$
$$\boldsymbol{\omega}_{\mathbf{c2}} = \dot{\boldsymbol{\omega}}_{\mathbf{c2}} = \mathbf{0}$$
$$\mathbf{f_{12}} = \begin{bmatrix} \Gamma_2 \\ f_{y,12} \\ f_{z,12} \end{bmatrix}$$
$$\mathbf{f_{23}} = \mathbf{0}$$

The Newton-Euler equation for segment 2 is:

$$\mathbf{f_{12}} - \mathbf{f_{23}} + m_2 \left(\mathbf{g} - \dot{\mathbf{v}}_{\mathbf{c2}}\right) = \mathbf{0}$$
$$\Rightarrow \begin{bmatrix} \Gamma_2 \\ f_{y,12} \\ f_{z,12} \end{bmatrix} + m_2 \left( \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} - \begin{bmatrix} \ddot{q}_2 \\ 0 \\ \ddot{q}_1 \end{bmatrix} \right) = \mathbf{0}$$
$$\Rightarrow \begin{cases} \Gamma_2 & = m_2 \ddot{q}_2 \\ f_{y,12} & = 0 \\ f_{z,12} & = m_2 (g + \ddot{q}_1) \end{cases}$$

3. Finally, we solve the systems of equations to deduce the dynamic model formed by the expression of the joint torques for each segment:

$$\begin{cases} \Gamma_1 & = (m_1 + m_2)(g + \ddot{q}_1) \\ \Gamma_2 & = m_2 \ddot{q}_2 \end{cases}$$
$$\Leftrightarrow \boldsymbol{\Gamma} = \begin{bmatrix} (m_1 + m_2)(g + \ddot{q}_1) \\ m_2 \ddot{q}_2 \end{bmatrix}$$

One can observe that the result is the same as in exercise 1!